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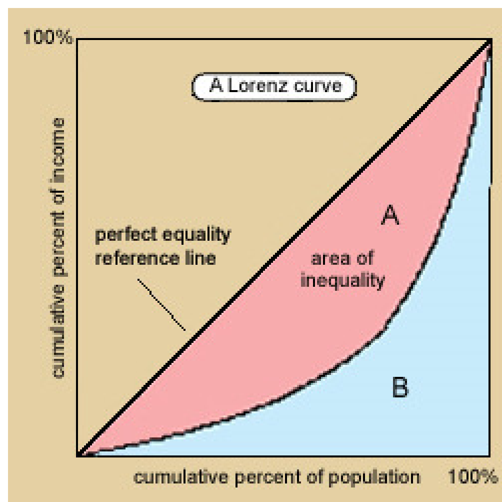
What is the distinction between positive and normative measures of income inequality? Refer to the properties of one positive and one normative measure. Can the Gini coefficient be interpreted as a normative measure of inequality? If so, is its underlying social welfare function justifiable? Is there always an implicit welfare function underlying positive welfare measures of inequality?

Positive measures of income inequality are primarily concerned with providing a descriptive explanation of income distribution. Derived from statistical concepts, positive measures make no explicit use of any concept of social welfare.[i] Examples of positive measures include the Gini coefficient, Lorenz curve, Theil measure, relative mean deviation, and the coefficient of variation. In contrast, normative measures of income inequality are based on an explicit formulation of social welfare and the loss incurred from unequal distribution.[ii] By linking and integrating the measure of inequality with social welfare, normative measures rely on value judgments and a properly defined welfare function.[iii] Examples of normative measures include the Dalton measure and the Atkinson index. This paper will first discuss the properties of the Lorenz curve and the Atkinson index, and then provide a broader discussion on the welfare implications of positive measures of inequality, using the Gini coefficient as an example.

Lorenz Curve:



As a function of the cumulative proportion of ordered individuals mapped onto the corresponding cumulative proportion of income share, the Lorenz cure is a positive measure used to represent income distribution.[iv] In effect, the Lorenz curve plots the cumulative percentage of total income received against the cumulative percentages of individuals ordered by income, thereby showing the y% of total income the bottom x% of households own.[v] The closer the Lorenz curve is to the 45 degree line, the more equal the income distribution. More formally, the Lorenz curve is represented by $L(y) = \frac{\int_0^y x dF(x)}{\mu}$, in which F(y) is the cumulative distribution function of ordered individuals and μ is average income. The slope of the Lorenz curve is x/μ, which is always positive for positive income. Likewise, the second derivative is $\frac{1}{\mu f(x)}$, also a positive number. Hence, the two derivatives imply that the slope of the Lorenz curve is positive and increases monotonically.[vi] The Gini coefficient, defined as the ratio between the area enclosed by the line of inequality and the Lorenz curve, and the total triangular area under the line of inequality, summarizes the degree of inequality. In the figure above, the Gini coefficient would be equal to: A/(A+B).[vii]

Atkinson Index:

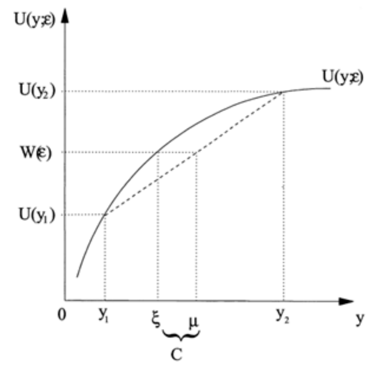
In contrast to the Lorenz Curve, which can be calculated statistically whenever income data is available, the Atkinson Index relies on a specific social welfare function. The underlying Atkinson additive social welfare function is given by $W(\varepsilon) = W(\rho = 1, \varepsilon) = \int_0^1 U(Q(p); \varepsilon))\, dp$, which can be interpreted as a utilitarian social welfare function, where $U(Q(p); \varepsilon)$ is an individual utility function with decreasing marginal utility of income. Notably, ρ – the “ethical parameter” representing concern for the "share deficits" at various cumulative proportions of the population – is assumed to be one.[viii] Hence, by setting ρ=1, the Atkinson index assumes that income ranks are not explicitly important in computing social welfare, in contrast to the general social welfare function.

Parameter ε represents aversion to inequality, with typical values ranging from 0.5 to 2. When ε approaches 0, the marginal social utility is constant – increasing a poor person's income has the same social welfare impact as increasing a rich person's income. Conversely, as ε rises, society attaches more weight to income transfers at the lower end of the distribution and less weight to transfers at the top (greater social desirability of increasing the income of the poor than the rich).[ix]

The Atkinson inequality index is represented by:

$$I(\varepsilon) = I(\rho = 1, \varepsilon) = 1 - \frac{\left(\int_0^1 Q(p)^{1-\varepsilon} dp\right)^{\frac{1}{1-\varepsilon}}}{\mu}, when\ \varepsilon \neq 1$$

$$I(\varepsilon) = I(\rho = 1, \varepsilon) = 1 - \frac{exp(\int_0^1 \ln\left(Q(p)\right) dp)}{\mu}, when\ \varepsilon = 1$$



The graph visually depicts the "cost of inequality" in relation to the Atkinson social welfare function, in the case of a population of two individuals with income y_1 and y_2 with mean income μ. The utility function has decreasing marginal utility, and W(ε) is given by the average of the utilities, $[U(y_1)+ U(y_2)]/2$. If income were equally distributed, then a mean income of ξ would be sufficient to generate the same level of social welfare. Hence, the cost of inequality, is C, the distance between μ and ξ, and the inequality measure I(ε) is represented by the ratio C/ μ. A higher ε increases the convexity of the utility function, resulting in a greater cost of inequality and a greater inequality index.[x]

Positive Welfare Measures and Implicit Social Welfare: The GINI coefficient as an example

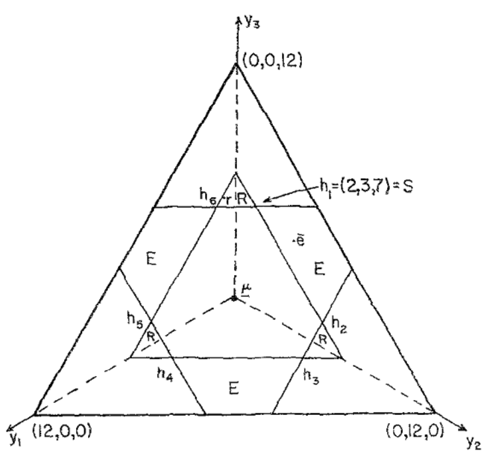
Dalton, Kolm, Atkinson, and Blockorby argue that every measure of inequality has some underlying notion of a social welfare function. [xi] As Dalton explains, unlike the biologist who is concerned with the measure of inequality in the distribution of a physical characteristic, an economist is primarily interested in the effects of income distribution on total economic welfare.[xii] Using different assumptions, economic theorists have proposed various proofs and algorithms for the derivation of the implied social welfare function for common positive measures of income inequality. This paper will focus on one such derivation, using the Gini coefficient as a basis of analysis.

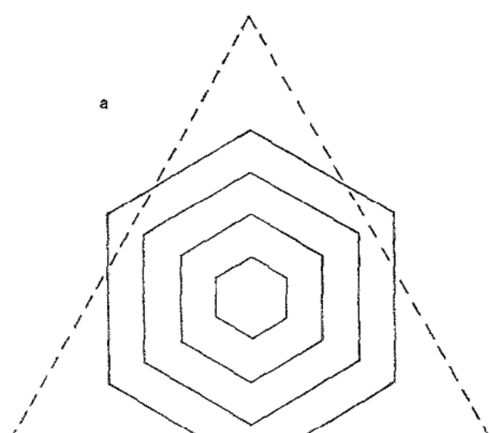
Using Malmquist's duality theory, Blockorby and Donaldson demonstrate that any social welfare function can generate a family of indices of relative inequality, and conversely, any family of indices can generate a family of social welfare functions. The general intuition of the proof is as follows. Relative index numbers compare two situations and are homogenous of degree zero in their arguments. Assuming that the indices are homogenous of degree one in the arguments of one situation and satisfy Fisher's circularity test, then each index can be written as a ratio whose numerator and denominator are the same linearly homogenous function evaluated at the values appropriate to the two situations. Hence, in the context of relative equality measures, one can derive an implicit representation of an ordinal social welfare function to imply the index. Using the theory of exact index numbers, in the special case that the social welfare function is homothetic, a *single* cardinally-significant index is implied. The authors then construct an algorithm that derives a social welfare function from any relative index.

Applying this method to the Gini coefficient and normalizing the index such that $\bar{E}$(μe)=1, the authors conclude that the Gini coefficient of relative inequality has the image $\bar{E}_g(y) = -\frac{1}{n} + \frac{2}{n^2\mu}(y_1 + 2y_2 + \cdots + iy_i + \cdots + ny_n$, where $y_1 \geq y_2 \ldots \geq y_n$. Likewise, the image of the Gini social welfare function can be represented as: $\bar{W}_g(y) = \frac{1}{n^2}(y_1 + 3y_2 + \cdots + (2i-1)y_i + \cdots + (2n-1)y_n$. Notably, the implied social welfare index of the Gini coefficient is quasi-concave and homothetic – additive, but not separable over Ω^n. In the simple three person case, the poorest person gets five times the weight of the richest person, and the intermediate person gets three times the weight of the richest person.[xiii]

Atkinson contends that statistical measures obscure the fact that a complete ranking of distributions cannot occur without the specification of a social welfare function, and that implicit social measures embedded in the conventional statistical measures have properties that may be unreasonable.[xiv] In the case of the Gini coefficient, the underlying social welfare function is distributionally homothetic, signifying that the way social welfare trades off income among individuals *is independent of how equal or unequal the distribution of income is*. Given that marginal rates of substitution are independent of scale, the Gini social welfare function ranks as indifferent some distributions of income that have negative components with others that have all positive components. As a result, the implied social welfare function by the Gini coefficient may have ethically perverse qualities.

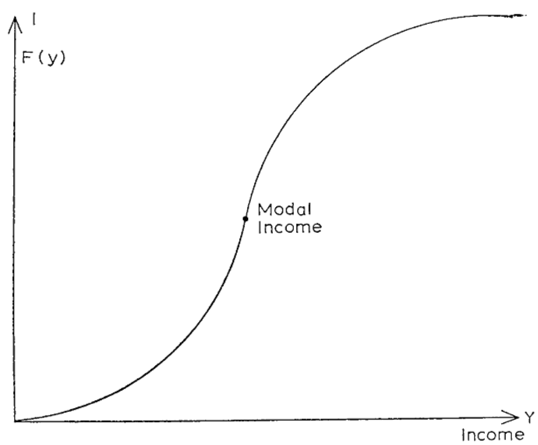
As a visual representation, compare the figure on the left depicting the Rawlsian min-max criterion with the figure on the right illustrating indifference surfaces of the implied social welfare function of the Gini coefficient. On the left, the status quo is represented by S, and the irregular hexagon formed by h1, h2, h3, h4, h5, and h6 constitute an indifference surface derived from any symmetric, quasi-concave social welfare function. Depending on the functional form of a specific social welfare function, parts of regions R and E will be ranked as indifferent to the status quo. Under a Rawlsian standard of justice, all distributions in regions R are ranked at least as good as the status quo (status of poorest improves, while intermediate person deteriorates), while distributions in region E are ranked worse (status of poorest person deteriorates, though the intermediate person improves). Blackorby and Donaldson then analyze the relative bias of various social welfare functions by comparing whether the indifference surface of a specific social welfare function distinguishes between R and E.





The figure on the right shows the indifference surfaces implied by the Gini social welfare function superimposed on the simplex to the left, attesting to the ethical problem of a distributionally homothetic functional form. In effect, the Gini social welfare function is indifferent to making the poorest worse off (even to negative territory) to the benefit of the intermediate.

Likewise, an examination by Atkinson of the underlying social welfare function of the Gini coefficient also reveals an unsatisfactory *relative* sensitivity to income transfers at different income levels. In essence, under a normal income distribution, the effect of an income transfer would be proportional to $F(y_1)-F(y_1-h)$[xv], clearly weighting transfers affecting middle income classes much more significantly than transfers affecting the poor. When gauging the impact of income inequality, this weighting does not fully accord with accepted normative beliefs.[xvi]



In conclusion, since nearly all positive inequality indexes can be decomposed into quasi-concave symmetric social welfare functions, they imply some set of social values and ethical standards. In the case of the Gini coefficient and other positive measures of income inequality, it is important to understand the implicit assumptions embedded in order to fully ascertain potential policy implications and to understand the limitations of each measure.

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[i] O'Hara, Phillip. Encyclopedia of Political Economy. Routledge. London, 1999.
[ii] Sen, Amartya. On Economic Inequality. Oxford University Press. Oxford, 1997.
[iii] Tinbergen, Jan. "A Positive and Normative Theory of Income Distribution." Review of Income and Wealth, vol. 16, issue 3, pages 221-34, 1970.
[iv] Damgaard, Christian. "Lorenz Curve." MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/LorenzCurve.html>
[v] "Glossary of statistical terms." OECD. <http://stats.oecd.org/glossary/detail.asp?ID=4843>
[vi] Kakwani, Nanak. Income Inequality and Poverty: Methods of Estimation and Policy Application. World Bank, 1980
[vii] "Lorenz Curves and Gini Coefficients." Economics Interactive, University of North Carolina. <http://www.unc.edu/depts/econ/byrns_web/Economicae/Figures/Lorenz.htm>
[viii] The general social welfare function is given by: $W = \int_0^1 U\big(Q(p)\big)\omega(p)dp$, where $\omega(p)$ is of the special form $\omega(p;\ \rho)$, defined as: $\omega(p;\ \rho) = \int_p^1 \kappa(q,p)dq = \rho(1-p)^{\rho-1}$. Hence, when ρ=1, then $\omega(p;\ \rho)$=1, indicating that the same weight is given to all deviations from the mean regardless of the income distribution under consideration.
[ix] MacArthur, Catherine. "Income Inequality." Research Network on Socioeconomic Status and Health. <http://www.macses.ucsf.edu/Research/Social%20Environment/notebook/inequality.html> June 2000.

[x] Silber, Jacques. Poverty and Equity: Measurement, Policy, and Estimation with Dad. Springer, 2006.
[xi] Blackorby, Charles, Donaldson, David. Measures of Relative Equality and Their Meaning in Terms of Social Welfare. Journal of Economic Theory. Vol 18, No 59-60, 1978.
[xii] Dalton, Hugh. "The Measurement of the Inequality of Incomes." The Economic Journal, Vol. 30, No. 119, pages 348-361. September 1920
[xiii] Blackorby, Charles, Donaldson, David. Measures of Relative Equality and Their Meaning in Terms of Social Welfare. Journal of Economic Theory. Vol 18, No 59-60, 1978.
[xiv] Atkinson, Anthony. "On the Measurement of Inequality." Journal of Economic Theory. Vol 2, 244-263, 1970.
[xv] The figure below exemplifies a typical income distribution, in which the perverse quality of the Gini coefficient would be evident.
[xvi] *Ibid*.